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Oscillations of Two-Dimensional Nonlinear Partial Difference Systems

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Abstract— This paper studies the following nonlinear two-dimensional partial difference system:

$$\Delta_1(x_{mn}) - b_{mn}g(y_{mn}) = 0,$$
$$T(\Delta_1, \Delta_2)(y_{mn}) + a_{mn}f(x_{mn}) = 0,$$

where $m, n \in N_i = \{i, i+1, \dots\}$, i is a nonnegative integer, $T(\Delta_1, \Delta_2) = \Delta_1 + \Delta_2 + I$, $\Delta_1 y_{mn} = y_{m+1,n} - y_{mn}$, $\Delta_2 y_{mn} = y_{m,n+1} - y_{mn}$, $I_{mn}y_{mn} = y_{mn}$, $\{a_{mn}\}$ and $\{b_{mn}\}$ are real sequences, $m, n \in N_0$, and $f, g : R \to R$ are continuous with $uf(u) > 0$ and $ug(u) > 0$ for all $u \neq 0$. A solution $(\{x_{mn}\}, \{y_{mn}\})$ of this system is oscillatory if both components are oscillatory. Some sufficient conditions are derived for all solutions of this system to be oscillatory.

Keywords— Nonlinear partial difference systems, Oscillation, Delay partial differential equations, Asymptotic behavior

1. INTRODUCTION

Of particular interest are the oscillatory behaviors of second-order nonlinear partial difference equations because they are discrete analogues of second-order nonlinear partial differential equations and they have many physical applications (see, for example, [1,2]). Recently, there has been an increasing interest in the study of the asymptotic behaviors of solutions, especially oscillatory behaviors, of delay partial difference equations (see, for example [3–10]).

It is an interesting question to ask if one can extend oscillatory criteria for second-order nonlinear partial differential equations to its discrete counterparts—nonlinear two-dimensional partial difference systems, and, if so, how. This paper attempts to provide some answer to this question.

More precisely, general nonlinear two-dimensional partial difference systems are considered in this paper, which include also half-linear and quasilinear partial difference equations as special cases.

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Consider the following nonlinear two-dimensional partial difference systems:

$$\begin{aligned} \Delta_1(x_{mn}) - b_{mn} g(y_{mn}) &= 0, \qquad m,n \in N_i = \{i, i+1, \dots\}, \\ T(\Delta_1, \Delta_2)(y_{mn}) + a_{mn} f(x_{mn}) &= 0, \qquad i = 0,1,2,\dots, \end{aligned} \tag{1}$$

where $T(\Delta_1, \Delta_2) = \Delta_1 + \Delta_2 + I$, $\Delta_1 y_{mn} = y_{m+1,n} - y_{mn}$, $\Delta_2 y_{mn} = y_{m,n+1} - y_{mn}$, $I_{mn} y_{mn} = y_{mn}$, $\{a_{mn}\}$ and $\{b_{mn}\}$ are real sequences, for $(m,n) \in N_0$, f, g are continuous real valued functions on R with the following sign property:

$$uf(u) > 0, \quad \text{and} \quad ug(u) > 0, \qquad \text{for } u \neq 0. \tag{2}$$

Throughout the paper, attention is restricted only to the solutions of system (1) that exist for $m, n \in N_0$. As usual, a real sequence defined on N_0 is said to be *oscillatory* if it is neither eventually positive nor eventually negative, and it is said to be *nonoscillatory* otherwise. A solution, $(\{x_{mn}\}, \{y_{mn}\})$, of system (1) is said to be *oscillatory* if both of its components are oscillatory, and it is said to be *nonoscillatory* otherwise. System (1) is said to be *oscillatory* if all its solutions are oscillatory.

Note that if $b_{mn} \geq 0$ and $b_{mn} \neq 0$ for infinitely many values of m, n, then it follows easily from the first equation in (1) that for any solution $(\{x_{mn}\}, \{y_{mn}\})$ of the system, the oscillation of $\{x_{mn}\}$ implies the oscillation of $\{y_{mn}\}$, as well. Thus, if $(\{x_{mn}\}, \{y_{mn}\})$ is a nonoscillatory solution of (1), then $\{x_{mn}\}$ is always nonoscillatory. Furthermore, if $a_{mn} \geq 0$ and $a_{mn} \neq 0$ for infinitely many m, n, then it follows from the second equation in (1) that $\{y_{mn}\}$ is an eventually one-sign sequence.

In the case where $b_{m,n} > 0$ for all $m, n \in N_0$, and $g(u) = u$ with $u \in R$, the difference system (1) reduces to the following second-order nonlinear difference equation:

$$T(\Delta_1, \Delta_2) \left\{ \frac{1}{b_{mn}} [\Delta_1(x_{mn})] \right\} + a_{mn} f(x_{mn}) = 0. \tag{3}$$

Furthermore, if $b_{mn} = 1$ for $m \geq m_0$, $n \geq n_0$, and $f(u) = |u|^\lambda \operatorname{sgn} u$, this equation becomes

$$T(\Delta_1, \Delta_2)[\Delta_1(x_{mn})] + a_{mn} |x_{mn}|^\lambda \operatorname{sgn} x_{mn} = 0. \tag{4}$$

Also, if $g(u) = u^\alpha$, where α is a ratio of odd positive integers, then system (1) reduces to the following quasilinear difference equation:

$$T(\Delta_1, \Delta_2) \left\{ \frac{1}{b_{mn}^{1/\alpha}} [\Delta_1(x_{mn})]^{1/\alpha} \right\} + a_{mn} f(x_{mn}) = 0. \tag{5}$$

Regarding oscillation criteria for (3), the reader is referred to [5]. It follows from (3)–(5) that system (1) has rich dynamics and, therefore, a careful study on this system may in turn yield useful information for its companion partial differential equations.

2. SOME PRELIMINARY LEMMAS

In this section, some useful preliminary results are first established, which will be needed for the proofs of the main results given in the next section.

The following notation will be used:

$$\sum_{i=v}^{u} *_i = 0, \qquad \text{for } u < v. \tag{6}$$

LEMMA 1. *(See [6].) For any sequence* $\{A_{mn}\}$

$$\sum_{i=m-k}^{m}\sum_{j=n-l}^{n}(A_{i+1,j}+A_{i,j+1}-A_{ij})=\sum_{i=m+1-k}^{m+1}\sum_{j=n+1-l}^{n}A_{ij}+\sum_{i=m-k}^{m}A_{i,n+1} - A_{m-k,n-l}+A_{m+1,n-l}.$$

LEMMA 2. *Suppose that* $w_{mn}\geq 0$. *Then,*

$$\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}B_{st}\Delta_1(w_{st})\geq B_{m-1,n-1}w_{m,n-1}-\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}b_{st}w_{s+1,t}, \tag{7}$$

where $B_{m,n}=\sum_{i=m_0}^{m-1}\sum_{j=n_0}^{n-1}b_{ij}$.

PROOF. First, note that

$$\begin{aligned}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}B_{st}\Delta_1(w_{st})&=\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\sum_{i=m_0}^{s-1}\sum_{j=n_0}^{t-1}b_{ij}(w_{s+1,t}-w_{st})\\&=\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left[\left(\sum_{i=m_0}^{s}\sum_{j=n_0}^{t}b_{ij}-\sum_{i=m_0}^{s-1}b_{it}-\sum_{j=n_0}^{t-1}b_{sj}-b_{st}\right)(w_{s+1,t}-w_{st})\right].\end{aligned} \tag{8}$$

Then, note that

$$\begin{aligned}\sum_{i=m_0}^{s}\sum_{j=n_0}^{t}b_{ij}&=\sum_{i=m_0}^{s-1}\sum_{j=n_0}^{t-1}b_{ij}+\sum_{j=n_0}^{t-1}b_{sj}+\sum_{i=m_0}^{s-1}b_{it}+b_{st}\\&\geq\sum_{i=m_0}^{s-1}\sum_{j=n_0}^{t-1}b_{ij}+\sum_{j=n_0}^{t-1}b_{sj}+\sum_{i=m_0}^{s-1}b_{it}.\end{aligned} \tag{9}$$

Substituting (9) into (8), and using (6), one obtains

$$\begin{aligned}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}B_{st}\Delta_1(w_{st})&\geq\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left(\sum_{i=m_0}^{s-1}\sum_{j=n_0}^{t-1}b_{ij}-b_{st}\right)(w_{s+1,t}-w_{st})\\&=\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left(\sum_{i=m_0}^{s-1}\sum_{j=n_0}^{t-1}b_{ij}w_{s+1,t}-\sum_{i=m_0}^{s-1}\sum_{j=n_0}^{t-1}b_{ij}w_{st}-b_{st}w_{s+1,t}+b_{st}w_{st}\right)\\&\geq\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left(\sum_{i=m_0}^{s-1}\sum_{j=n_0}^{t-1}b_{ij}w_{s+1,t}-\sum_{i=m_0}^{s-1}\sum_{j=n_0}^{t-1}b_{ij}w_{st}-b_{st}w_{s+1,t}\right)\\&=\sum_{s=m_0}^{m-1}\left(\sum_{t=n_0}^{n-1}\sum_{i=m_0}^{s-1}\sum_{j=n_0}^{t-1}b_{ij}w_{s+1,t}-\sum_{t=n_0}^{n-1}\sum_{i=m_0}^{s-1}\sum_{j=n_0}^{t-1}b_{ij}w_{st}\right)-\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}b_{st}w_{s+1,t}\\&\geq\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}b_{ij}w_{m,n-1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}b_{st}w_{s+1,t}\\&=B_{m-1,n-1}w_{m,n-1}-\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}b_{st}w_{s+1,t}.\end{aligned} \tag{10}$$

This implies that

$$\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}B_{st}\Delta_1(w_{st})\geq B_{m-1,n-1}w_{m,n-1}-\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}b_{st}w_{s+1,t}, \tag{11}$$

so that (7) is obtained. ∎

3. MAIN RESULTS

In this section, the above preliminary results are applied and generalized to system (1). The following conditions will be assumed and utilized.

(c_1) $a_{mn} \geq 0$ and $b_{mn} \geq 0$ for all $m, n \in N_0$, and neither sequence vanishes identically for $m, n \in N_0$.

(c_2) $\lim_{m,n\to\infty} B_{mn} = \infty$, where $B_{mn} = \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b_{st}$.

(c_3) $f(uv) \geq f(u)f(v)$, for all $u \geq 0$, $v \geq 0$.

(c_4) $\int_0^{\pm\alpha} ((du)/(f(g(u))) < \infty$, for all $\alpha > 0$.

(c_5) $\sum_{m=m_0}^{\infty} \sum_{n=n_0}^{\infty} f(1/(n-n_0)) f(B_{mn}) a_{mn} = \infty$.

(c_6) f and g are nondecreasing.

(c_7) Taking boundary conditions $x_{m_0 n} = c_n \geq 0$ and $x_{m n_0} = d_m \geq 0$ such that $\sum_{n=n_0}^{\infty} c_n < \infty$ and $\sum_{m=m_0}^{\infty} d_m < \infty$.

(c_8) There exist nonnegative functions g_1 and f_1 such that $f(u) - f(v) = (u-v) f_1(u,v)$, $g(u) - g(v) = (u-v) g_1(u,v)$, and $g_1(u,v) \geq \xi > 0$, for $u, v \neq 0$.

(c_9) $\int_{\pm\alpha}^{\pm\infty} ((du)/(f(u)) < \infty$, for every $\alpha > 0$.

(c_{10}) $\sum_{m=m_0}^{\infty} \sum_{n=n_0}^{\infty} B_{mn} a_{mn} = \infty$.

THEOREM 1. *Suppose that Conditions (c_1)–(c_7) and (2) are satisfied. Then, system (1) is oscillatory.*

PROOF. Suppose that system (1) has a nonoscillatory solution $(\{x_{mn}\}, \{y_{mn}\})$, for $m, n \in N_0$. Since $\{a_{mn}\}$ and $\{b_{mn}\}$ are not identically zero for $m, n \in N_0$, as noted earlier, $\{x_{mn}\}$ and $\{y_{mn}\}$ are eventually of one sign. So, without loss of generality, assume that $x_{mn} \neq 0$ for all $m, n \in N_0$. Furthermore, observe that the substitutions $z_{mn} = -x_{mn}$ and $w_{mn} = -y_{mn}$ transform system (1) into the following:

$$\begin{aligned} \Delta_1(z_{mn}) &= b_{mn} \bar{g}(w_{mn}), \\ T(\Delta_1, \Delta_2)(w_{mn}) &= -a_{mn} \bar{f}(z_{mn}), \qquad m, n \in N_0, \end{aligned} \tag{12}$$

where $\bar{f}(u) = -f(u)$, $u \in R$, and $\bar{g}(v) = -g(v)$, $v \in R$. So, functions $\bar{f}$ and $\bar{g}$ are subjected to the condition imposed on f and g, respectively. For this reason, discussion is restricted only to the case where $\{x_{mn}\}$ is eventually positive for $m \geq m_0$, $n \geq n_0$ and either $y_{mn} \geq 0$ or $y_{mn} \leq 0$, for $m \geq m_0$, $n \geq n_0$.

First, consider the case of $y_{mn} \leq 0$ for $m \geq m_0$, $n \geq n_0$. The second equation in (1) implies that $\{y_{mn}\}$ is decreasing. Since $y_{mn} \leq 0$, it approaches either $-\infty$ or a finite negative value as $m, n \to \infty$. Likewise, $g(y_{mn}) \to -\infty$ or to a finite negative value as $m, n \to \infty$. This, in view of Condition (c_2), implies that

$$\sum_{m=n_0}^{\infty} \sum_{n=n_0}^{\infty} b_{mn} g(y_{mn}) = -\infty. \tag{13}$$

Now, taking m, n sufficiently large, summing the first equation on both sides of (1) from $m-1$, $n \to \infty$, one finds that

$$\sum_{j=n_0}^{n-1} (x_{mj} - x_{m_0 j}) = \sum_{i=m_0}^{m-1} \sum_{j=n_0}^{n-1} (x_{i+1,j} - x_{ij}) = \sum_{i=m_0}^{m-1} \sum_{j=n_0}^{n-1} \Delta_1(x_{mn}) = \sum_{i=m_0}^{m-1} \sum_{j=n_0}^{n-1} b_{ij} g(y_{ij}),$$

so that

$$x_{m,n-1} \leq \sum_{j=n_0}^{n-1} x_{mj} = \sum_{j=n_0}^{n-1} x_{m_0 j} + \sum_{i=m_0}^{m-1} \sum_{j=n_0}^{n-1} b_{ij} g(y_{ij}). \tag{14}$$

Using (c_7) and (14), one obtains

$$x_{m,n-1} \leq \sum_{j=n_0}^{n-1} x_{m_0 j} + \sum_{i=m_0}^{m-1} \sum_{j=n_0}^{n-1} b_{ij} g(y_{ij}) \to -\infty, \qquad \text{as } m, n \to \infty,$$

which is a contradiction to the assumption that $x_{mn} > 0$ for all $m \geq m_0$, $n \geq n_0$.

Next, consider the case where $y_{mn} \geq 0$, for all $m \geq m_0$, $n \geq n_0$. From system (1), it is clear that $\{x_{mn}\}$ is increasing in m and $\{y_{mn}\}$ is decreasing in m, n. Thus, applying (c$_6$),(c$_7$) and summing the first equation in (1) yield

$$\begin{aligned}(n-n_0)x_{m,n-1} - \sum_{j=n_0}^{n-1} x_{m_0 j} \geq \sum_{j=n_0}^{n-1} x_{mj} - \sum_{j=n_0}^{n-1} x_{m_0 j} &= \sum_{j=n_0}^{n-1} (x_{mj} - x_{m_0 j}) \\ &= \sum_{i=m_0}^{m-1} \sum_{j=n_0}^{n-1} (x_{i+1,j} - x_{ij}) = \sum_{i=m_0}^{m-1} \sum_{j=n_0}^{n-1} b_{ij} g(y_{ij}).\end{aligned}$$

Hence,

$$\begin{aligned} x_{m,n-1} &\geq \frac{1}{(n-n_0)} \left[\sum_{j=n_0}^{n-1} x_{m_0 j} + \sum_{i=m_0}^{m-1} \sum_{j=n_0}^{n-1} b_{ij} g(y_{ij}) \right] \\ &\geq \frac{1}{(n-n_0)} \left[\sum_{i=m_0}^{m-1} \sum_{j=n_0}^{n-1} b_{ij} g(y_{ij}) \right] \\ &\geq \frac{1}{(n-n_0)} \left[\sum_{i=m_0}^{m-1} \sum_{j=n_0}^{n-1} b_{ij} g(y_{m-1,n-1}) \right] \\ &= \frac{1}{(n-n_0)} B_{mn} g(y_{m-1,n-1}) \geq \frac{1}{(n-n_0)} B_{mn} g(y_{mn}). \end{aligned} \tag{15}$$

Next, one can show that

$$\Delta_2(x_{mn}) \geq 0, \qquad \text{for } m \geq m_0, \quad n \geq n_0. \tag{16}$$

In fact, if there would exist $m_1 \geq m_0$ and $n_1 \geq n_0$ such that $\Delta_2(x_{m_1 n_1}) = c < 0$, then $\Delta_2(x_{mn}) \leq c$ for $m \geq m_1$, $n \geq n_1$, so that

$$\sum_{i=m_1+1}^{m} (x_{im} - x_{in_1}) = \sum_{i=m_1+1}^{m} \sum_{j=n_1+1}^{n} \Delta_2(x_{ij}) \leq \sum_{i=m_1+1}^{m} \sum_{j=n_1+1}^{n} c = c(m-m_1)(n-n_1).$$

Note that, (c$_7$) yields

$$x_{mn} \leq \sum_{i=m_2+1}^{m} x_{in} \leq \sum_{i=m_2+1}^{m} x_{in_1} + c(m-m_1)(n-n_1) \to -\infty, \qquad \text{as } m,n \to \infty,$$

which contradicts the fact that $x_{mn} > 0$, for $m \geq m_1$, $n \geq n_1$. Thus, (16) holds, that is, $\{x_{mn}\}$ is also increasing in n. Therefore, it follows from (15) that

$$x_{mn} \geq x_{m,n-1} \geq \frac{1}{(n-n_0)} B_{mn} g(y_{mn}).$$

In view of (c$_2$), for all m, n large enough, $B_{mn} \geq 1$, so

$$f(x_{mn}) \geq f\left(\frac{1}{(n-n_0)} B_{mn}\right) f(g(y_{mn})) \geq \left(\frac{1}{(n-n_0)}\right) f(B_{mn}) f(g(y_{mn})).$$

The second equation in (1) leads to

$$T(\Delta_1, \Delta_2)(y_{mn}) \leq -a_{mn} f\left(\frac{1}{(n-n_0)}\right) f(B_{mn}) f(g(y_{mn})),$$

that is,

$$\frac{T(\Delta_1,\Delta_2)(y_{mn})}{f(g(y_{mn}))} \le -a_{mn} f\left(\frac{1}{(n-n_0)}\right) f(B_{mn}). \tag{17}$$

Moreover, using (17), we have

$$\begin{aligned}
-\frac{\Delta_1(y_{mn})}{f(g(y_{mn}))} - \frac{\Delta_2(y_{mn})}{f(g(y_{mn}))} &= -\frac{\Delta_1(y_{mn})+\Delta_2(y_{mn})}{f(g(y_{mn}))} = -\frac{(y_{m+1,n}-y_{mn})+(y_{m,n+1}-y_{mn})}{f(g(y_{mn}))} \\
&= -\frac{y_{m+1,n}+y_{m,n+1}-2y_{mn}}{f(g(y_{mn}))} \ge -\frac{y_{m+1,n}+y_{m,n+1}-y_{mn}}{f(g(y_{mn}))} \\
&= -\frac{T(\Delta_1,\Delta_2)(y_{mn})}{f(g((y_{mn}))} \ge a_{mn} f\left(\frac{1}{(n-n_0)}\right) f(B_{mn}),
\end{aligned}$$

that is,

$$-\frac{\Delta_1(y_{mn})}{f(g(y_{mn}))} - \frac{\Delta_2(y_{mn})}{f(g(y_{mn}))} \ge a_{mn} f\left(\frac{1}{(n-n_0)}\right) f(B_{mn}). \tag{}$$

Summing the inequality on both sides of (18) from $m_0, n_0 \to m-1, n-1$, one obtains

$$-\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} \frac{\Delta_1(y_{st})}{f(g(y_{st}))} - \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} \frac{\Delta_2(y_{st})}{f(g(y_{st}))} \ge \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a_{st} f\left(\frac{1}{(t-n_0)}\right) f(B_{st}). \tag{19}$$

Observe that for $y_{m+1,n} \le s \le y_{mn}$, one has $f(g(y_{mn})) \ge f(g(s))$, that is,

$$\frac{1}{f(g(s))} \ge \frac{1}{f(g(y_{mn}))}.$$

Therefore,

$$-\frac{\Delta_1(y_{mn})}{f(g(y_{mn}))} \le \int_{y_{m+1,n+1}}^{y_{mn}} \frac{ds}{f(g(s))}. \tag{20}$$

Similarly,

$$-\frac{\Delta_2(y_{mn})}{f(g(y_{mn}))} \le \int_{y_{m+1,n+1}}^{y_{mn}} \frac{dt}{f(g(t))}. \tag{21}$$

It then follows from (20) and (21) that

$$-\frac{\Delta_1(y_{mn})}{f(g(y_{mn}))} - \frac{\Delta_2(y_{mn})}{f(g(y_{mn}))} \le \int_{y_{m+1,n+1}}^{y_{mn}} \frac{ds}{f(g(s))} + \int_{y_{m+1,n+1}}^{y_{mn}} \frac{dt}{f(g(t))}. \tag{22}$$

Summing the inequality on both sides of (22) from $m_0, n_0 \to m-1, n-1$, one has

$$\begin{aligned}
-\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} \frac{\Delta_1(y_{st})}{f(g(y_{st}))} - \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} \frac{\Delta_2(y_{st})}{f(g(y_{st}))} &\le \sum_{t=n_0}^{n-1}\sum_{s=m_0}^{m-1} \int_{y_{s+1,t+1}}^{y_{st}} \frac{ds}{f(g(s))} \\
&\quad + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} \int_{y_{s+1,t+1}}^{y_{st}} \frac{dt}{f(g(t))} \\
&= 2\int_{y_{mn}}^{y_{m_0 n_0}} \frac{ds}{f(g(s))}.
\end{aligned} \tag{23}$$

Now, combining (19) and (23) yields

$$\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a_{st} f\left(\frac{1}{(t-n_0)}\right) f(B_{st}) \le 2\int_{y_{mn}}^{y_{m_0 n_0}} \frac{ds}{f(g(s))},$$

which, in view of Conditions (c_4) and (c_5), leads to a contradiction. This completes the proof of the theorem. ∎

THEOREM 2. *Suppose that Conditions (c_1), (c_4), and (c_8)–(c_{10}) are satisfied. Then, system (1) is oscillatory.*

PROOF. Suppose, on the contrary, that system (1) is not oscillatory, so that $\{x_{mn}\}$ and $\{y_{mn}\}$ are of one sign for $m, n \in N_0$. Without loss of generality, assume that $x_{mn} \geq 0$. The case $y_{mn} \leq 0$ can be handled exactly as it was in the proof of Theorem 1.

Now, consider the case where $y_{mn} \geq 0$ for all $m \geq m_0$, $n \geq n_0$. Define

$$w_{mn} = \frac{g(y_{mn})}{f(x_{mn})}.$$

Applying system (1), inequality (16), and Condition (c_4), one has

$$f(x_{m+1,n}) \geq f(x_{mn}) \qquad \text{and} \qquad g(y_{m+1,n}) \geq g(y_{mn}). \tag{24}$$

Moreover, note that

$$\begin{aligned} \Delta_1(w_{mn}) = w_{m+1,n} - w_{mn} &= \left[\frac{g(y_{m+1,n})}{f(x_{m+1,n})} - \frac{g(y_{mn})}{f(x_{mn})}\right] \\ &= \frac{f(x_{mn})[g(y_{m+1,n}) - g(y_{mn})] - g(y_{mn})[f(x_{m+1,n}) - f(x_{mn})]}{f(x_{m+1,n})f(x_{mn})}. \end{aligned} \tag{25}$$

Combining (c_8) and the second equation in system (1) gives

$$\begin{aligned} g(y_{m+1,n}) - g(y_{mn}) &= (y_{m+1,n} - y_{mn})g_1(y_{m+1,n}, y_{mn}) \\ &\leq (y_{m+1,n} + y_{m,n+1} - y_{mn})g_1(y_{m+1,n}, y_{mn}) \\ &= -a_{mn}f(x_{mn})g_1(y_{m+1,n}, y_{mn}). \end{aligned} \tag{26}$$

Similarly,

$$f(x_{m+1,n}) - f(x_{mn}) \leq b_{mn}g(y_{mn})f_1(x_{m+1,n}, x_{mn}). \tag{27}$$

Substituting (26) and (27) into (25) gives

$$\begin{aligned} \Delta_1(w_{mn}) &\leq \frac{-a_{mn}f^2(x_{mn})g_1(y_{m+1,n}, y_{mn}) - b_{mn}g^2(y_{mn})f_1(x_{m+1,n}, x_{mn})}{f(x_{m+1,n})f(x_{mn})} \\ &= -a_{mn}\frac{f(x_{mn})}{f(x_{m+1,n})}g_1(y_{m+1,n}, y_{mn}) - \frac{b_{mn}g^2(y_{mn})f_1(x_{m+1,n}, x_{mn})}{2f(x_{m+1,n})f(x_{mn})} \\ &\leq -a_{mn}\left[\frac{f(x_{mn})}{f(x_{m+1,n})}\right]g_1(y_{m+1,n}, y_{mn}). \end{aligned} \tag{28}$$

Using (24), and in view of (28), one obtains

$$\Delta_1(w_{mn}) \leq -a_{mn}g_1(y_{m+1,n}, y_{mn}) \leq -a_{mn}\xi.$$

Multiplying both sides of this inequality by B_{mn}, summing from m_0, n_0 to $m-1$, $n-1$, and then applying Lemma 2, one obtains

$$B_{m-1,n-1}w_{m,n-1} - \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b_{st}w_{s+1,t} \leq -\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} \xi B_{st}a_{st}. \tag{29}$$

It follows from (24) and definition of w_{mn} that w_{mn} is decreasing in m, n, so that, in view of (29), one has

$$B_{m-1,n-1}w_{mn} - \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b_{st}w_{s+1,t} \leq -\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} \xi B_{st}a_{st}. \tag{30}$$

In view of Condition (c_{10}) and the positivity of $b_{mn}w_{mn}$, to complete the proof of the theorem, it suffices to show that the second term on the left-hand side of (30) is bounded.

Using the first equation in (1) and also (24), it is clear that

$$\begin{aligned} \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b_{st} w_{s+1,t} &= \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b_{st} \frac{g(y_{s+1,t})}{f(x_{s+1,t})} \le \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b_{st} \frac{g(y_{st})}{f(x_{s+1,t})} \\ &= \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} \frac{\Delta_1(x_{st})}{f(x_{s+1,t})}. \end{aligned} \tag{31}$$

Observe that for $x_{mn} \le s \le x_{m+1,n}$, one has $1/f(s) \ge 1/f(x_{m+1,n})$, and, therefore,

$$\int_{x_{mn}}^{x_{m+1,n+1}} \frac{ds}{f(s)} \ge \frac{\Delta_1(x_{st})}{f(x_{s+1,t})}. \tag{32}$$

Summing both sides of (32) from $m_0, n_0 \to m-1, n-1$, one obtains

$$\int_{x_{m_0 n_0}}^{x_{mn}} \frac{ds}{f(s)} = \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} \int_{x_{st}}^{x_{s+1,t+1}} \frac{ds}{f(s)} \ge \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} \frac{\Delta_1(x_{st})}{f(x_{s+1,t})}. \tag{33}$$

Combining (31) and (33), one finally obtains

$$\infty > \int_{x_{m_0 n_0}}^{x_{mn}} \frac{ds}{f(s)} \ge \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b_{st} w_{s+1,t}.$$

This completes the proof of the theorem. ∎

4. REMARKS AND EXAMPLES

One may similarly consider the following system:

$$\begin{aligned} \Delta_1(x_{mn}) - b_{mn} g(y_{m+1,n}) &= 0, \qquad (m,n) \in N_i = \{i, i+1, \dots\}, \\ T(\Delta_1, \Delta_2)(y_{mn}) + a_{mn} f(x_{mn}) &= 0, \qquad i = 0, 1, 2, \dots. \end{aligned} \tag{34}$$

Theorems 1 and 2 also hold for system (34). Due to limitation of space, details are omitted here.

Note also that the following systems can also be studied in a similar way:

$$\begin{aligned} \Delta_2(x_{mn}) - b_{mn} g(y_{mn}) &= 0, \qquad (m,n) \in N_i = \{i, i+1, \dots\}, \\ T(\Delta_1, \Delta_2)(y_{mn}) + a_{mn} f(x_{mn}) &= 0, \qquad i = 0, 1, 2, \dots, \end{aligned}$$

and

$$\begin{aligned} \Delta_2(x_{mn}) - b_{mn} g(y_{m+1,n}) &= 0, \qquad (m,n) \in N_i = \{i, i+1, \dots\}, \\ T(\Delta_1, \Delta_2)(y_{mn}) + a_{mn} f(x_{mn}) &= 0, \qquad i = 0, 1, 2, \dots, \end{aligned}$$

where $\Delta_2(x_{mn}) = x_{m,n+1} - x_{mn}$, for which similar results can be obtained.

Two examples are finally provided for illustration of the theoretical results derived in this paper.

EXAMPLE 1. Consider the partial difference system

$$\begin{aligned} \Delta_1(x_{mn}) - 2(m+n+1)^{1/3} y_{mn}^{1/3} &= 0, \\ T(\Delta_1, \Delta_2)(y_{mn}) + \frac{3(m+n)+4}{(m+n+2)(m+n+1)} x_{mn}^{5/3} &= 0, \end{aligned} \tag{35}$$

where

$$a_{mn} = \frac{3(m+n)+4}{(m+n+2)(m+n+1)}, \qquad b_{mn} = 2(m+n+1)^{1/3}, \qquad g(u) = u^{1/3}, \qquad f(v) = v^{5/3}.$$

All conditions of Theorem 1 are satisfied and, hence, system (35) is oscillatory. In fact,

$$(\{x_{mn}\}, \{y_{mn}\}) = \left(\{(-1)^{m+n}\}, \left\{\frac{(-1)^{m+n+1}}{m+n+1}\right\}\right)$$

is such an oscillatory solution of this system.

EXAMPLE 2. Consider the partial difference system

$$\begin{aligned} \Delta_1(x_{mn}) - \frac{2}{5^n(5^{2n}+1)}\left(y_{mn}^3 + y_{mn}\right) &= 0, \\ T(\Delta_1, \Delta_2)(y_{mn}) + 3x_{mn}^3 &= 0, \end{aligned} \tag{36}$$

where $(m,n) \in N_0$, $a_{mn} = 3$, $b_{mn} = 2/(5^n(5^{2n}+1))$, $g(u) = u^3+u$, and $f(v) = v^3$. All conditions of Theorem 2 are satisfied and, hence, system (36) is oscillatory. Indeed, $(\{x_{mn}\}, \{y_{mn}\}) = (\{(-1)^m\}, \{(-1)^{m+1} \times 5^n\})$ is such an oscillatory solution of this system.

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